Linearized slip flow past a semi-infinite flat plate

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Incompressible slip flow past a semi-infinite flat plate at zero incidence is treated in terms of the linearized viscous flow equations. A formal solution is obtained using Fourier transforms and the Wiener–Hopf technique. Explicit inversion of the transform is not possible, but asymptotic expansions are discussed. These reveal the inadequacy of boundary-layer theory in predicting the nature of the solution, even at the plate surface. For example, the local shear forces on the plate are significantly different from boundary-layer values, even far downstream, where slip effects are small. The boundary-layer limit is approached as the Reynolds number based on the mean free path or, equivalently, the free-stream Mach number tends to infinity.

1. Introduction

The first estimate for the effect of slip on the flow over a flat plate was made in 1949 by Donaldson. Since then numerous refinements and extensions, of which the present paper is one, have been made. Unfortunately, from the point of view of the theoretician working on the subject, experimental confirmation of the predictions of these analyses has not been forthcoming. Thus, it appears today that in the most important cases of practical interest, namely, in hypersonic low-density flows, slip effects are masked by shock-wave–boundary-layer interactions effects and it is even claimed (Probstein 1960) that slip effects may never be significant in such cases. In spite of this, the more academic interest in the phenomenon of slip flow persists and it seems desirable to be able to treat the low-speed incompressible case more carefully, for it is in this régime that conclusive comparisons of theory and experiment can be made.

The present analysis, therefore, is an attempt to solve completely the slip-flow problem for incompressible flow over a flat plate. No restriction is placed on the magnitude of the slip velocity

$$u' = a_1 \lambda \left| \frac{\partial u}{\partial y} \right|$$

at the plate surface, $y = 0$. Here $\partial u/\partial y$ is the normal velocity gradient, λ is the mean free path and a_1 is a constant. This boundary condition on the surface velocity, as derived from kinetic theory, holds strictly only for small departures from continuum flow, i.e. for small values of the slip velocity u', and most work has been restricted to the study of small perturbations about zero-slip boundary-layer solutions (Schaaf & Chambré 1957). The possibility exists, however, that the slip condition has an extended range of validity (Laurmann 1958; Sherman & Talbot 1958; Yang & Lees 1956), holding not only for small values of the mean free path λ, but also for arbitrary large departures from no-slip flow, and it is this viewpoint that provides the interest in the type of solution considered here. Indeed, the need for the more general approach is demonstrated by the result, derived below, that the low-speed case cannot be handled correctly in terms of boundary-layer theory, even far downstream where slip effects become relatively small.

In view of these observations it would clearly be desirable to analyse the slip-flow problem using the full Navier–Stokes equations. However, since these equations have not yet been solved even for the simpler zero-slip case, we will be satisfied here to treat the problem in terms of the linearized Oseen model, which we anticipate will contain most of the essential features of the full solution.

2. Formulation of the problem

We consider the two-dimensional incompressible Oseen equations

$$U\Delta\frac{\partial u}{\partial x} = \nu\Delta^2 u, \quad U\Delta\frac{\partial v}{\partial y} = \nu\Delta^2 v, \quad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0 \tag{2.1}$$

in which Δ is the Laplacian operator $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$; U is the free-stream velocity, directed along the x-axis; u and v are perturbations from this velocity in the x- and y-directions, respectively; and ν is the kinematic viscosity. The boundary conditions are taken as

$$u, v \to 0, \quad y \to \infty,$$

and

$$\left.\begin{aligned} u+U &= a_1\lambda\left|\frac{\partial u}{\partial y}\right|, & x>0,\\ \frac{\partial u}{\partial y} &= 0, & x<0,\\ v &= 0, & \end{aligned}\right\} \quad y=0, \tag{2.2}$$

where λ is the mean free path and the coefficient a_1 in the slip-boundary condition is a numerical constant whose value depends on the plate reflexion coefficient σ:

$$a_1 = (2-\sigma)/\sigma, \tag{2.3}$$

so that, for perfectly diffuse reflexion, $a_1 = 1$.

To solve the system of equations (2.1), under boundary conditions (2.2), we construct the general solution in terms of an integral of the fundamental solution† of (2.1). A detailed discussion of the fundamental solution of the linearized flow equations has been given by Lagerstrom, Cole & Trilling (1949). They find that it can be expressed as the sum of three terms,

$$\mathbf{v}_0 = \mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_2^* \tag{2.4}$$

† Given a differential form $L[u] \equiv \Sigma A_{ij}(\partial^2 u/\partial x_j\partial x_i)$, the function $K(x, y; \xi, \eta)$ such that $L[K(x, y; \xi, \eta)] = \delta(x-\xi, y-\eta)$, where δ is the Dirac delta function, is called the *fundamental solution* of $L[u]$.

where, in the incompressible case,

$$\left.\begin{aligned} \mathbf{v}_1 &= \frac{1}{2\pi U}\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}\ln r, \quad \mathbf{v}_2 = -\frac{1}{2\pi\nu}\exp\left(\frac{Ux}{2\nu}\right)K_0\left(\frac{Ur}{2\nu}\right)\begin{pmatrix}1\\0\end{pmatrix},\\ \mathbf{v}_2^* &= \frac{1}{2\pi U}\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}\left[\exp\left(\frac{Ux}{2\nu}\right)K_0\left(\frac{Ur}{2\nu}\right)\right].\end{aligned}\right\} \tag{2.5}$$

Here $r = (x^2+y^2)^{\frac{1}{2}}$ and K_0 is the modified Bessel function of zero order; $\mathbf{v}_1$ is the velocity of the so-called longitudinal wave and satisfies Laplace's equation. In the sense of Lagerstrom & Cole (1955), it is the 'outer' solution for the flow. The 'inner' part, equal to $\mathbf{v}_2+\mathbf{v}_2^*$, is the solenoidal 'transverse wave' and includes all the viscous effects. However, as shown by Lagerstrom *et al.* (1949), $\mathbf{v}_2^*$ is irrotational and the vorticity is given entirely by $\mathbf{v}_2$. Thus the latter component corresponds to the boundary layer, or at least to the vorticity boundary layer. However, in general, and in particular for our problem near or at the leading edge, significant contributions to $\mathbf{v}_0$ are made by all three components and it is not possible to represent the flow field close to the body by only the transverse wave or a part of it.

We now propose as the general solution

$$\mathbf{v} = \int_0^\infty \mathbf{v}_0(x-t,y)f(t)\,dt, \tag{2.6}$$

where $f(t)$ is an unknown distribution function. For convenience we define

$$F(t) = \exp(-Ut/2\nu)f(t). \tag{2.7}$$

From the properties of fundamental solutions, it follows that, when $y = 0$,

$$v = 0, \qquad \frac{\partial u}{\partial y} = 0 \quad (x < 0), \qquad \frac{\partial u}{\partial y} = \pm\frac{U}{2\nu}f(x) \quad (x > 0); \tag{2.8}$$

in the last equation the positive sign holds for $y \to 0$ from above and the negative sign for $y \to 0$ from below. Comparing (2.2) and (2.8) we see that all the boundary conditions are satisfied automatically by (2.6), except for the slip condition, which requires that

$$u+U = a_1(\lambda U/2\nu)f(x) \quad (y = 0). \tag{2.9}$$

If we now combine (2.6) and (2.9), we obtain

$$\begin{aligned} U+\frac{1}{2\pi}\exp\left(\frac{Ux}{2\nu}\right)\int_0^\infty F(t)\left[\frac{1}{x-t}\exp\left\{-\frac{U(x-t)}{2\nu}\right\}-\frac{U}{2\nu}K_0\left(\frac{U|x-t|}{2\nu}\right)\right.\\ \left.-\operatorname{sgn}(x-t)\frac{U}{2\nu}K_1\left(\frac{U|x-t|}{2\nu}\right)\right]dt = a_1\frac{\lambda U}{2\nu}F(x)\exp\left(\frac{Ux}{2\nu}\right) \quad (x > 0), \end{aligned} \tag{2.10}$$

which is an integral equation for the determination of $F(t)$.

3. Transformation of the integral equation

In §2 the problem was reduced to the solution of the integral equation (2.10). This equation is of the Wiener–Hopf type and there are standard methods of solution based on the use of the Fourier transform (Noble 1958).

Study of the equation shows that, in order to obtain existence of the Fourier transforms in a finite strip of the transformed w-plane, we need to introduce a convergence factor $\exp(-\alpha|x-t|)$, α constant, in the first (potential) term of the integral. After the nature of the solution in the w-plane has been studied, we can let $\alpha \to 0$. Thus, we write

$$U\exp\left(-\frac{Ux}{2\nu}\right)+\frac{1}{2\pi}\int_0^\infty F(t)\left[\frac{1}{x-t}\exp\left\{-\frac{U(x-t)}{2\nu}-\alpha|x-t|\right\}\right.$$
$$\left.-\frac{U}{2\nu}K_0\left(\frac{U|x-t|}{2\nu}\right)-\frac{U}{2\nu}\operatorname{sgn}(x-t)\,K_1\left(\frac{U|x-t|}{2\nu}\right)\right]dt=\frac{a_1\lambda U}{2\nu}F(x)\quad(x>0). \tag{3.1}$$

In order to be able to use the convolution theorem for Fourier transforms, we must modify (3.1) to hold for all x. Hence we introduce

$$u_-(x)=\begin{cases}u(x)\exp\left(\dfrac{Ux}{2\nu}\right) & (x<0);\\ 0 & (x>0);\end{cases} \tag{3.2}$$

and define

$$F_+(x)=\begin{cases}F(x) & (x>0);\\ 0 & (x<0);\end{cases}\qquad U_+(x)=\begin{cases}U & (x>0);\\ 0 & (x<0).\end{cases} \tag{3.3}$$

Then (3.1) becomes

$$U_+\exp\left(-\frac{Ux}{2\nu}\right)+\frac{1}{2\pi}\int_{-\infty}^\infty F_+(t)\left[\frac{1}{x-t}\exp\left\{-\frac{U(x-t)}{2\nu}-\alpha|x-t|\right\}\right.$$
$$\left.-\frac{U}{2\nu}K_0\left(\frac{U|x-t|}{2\nu}\right)-\frac{U}{2\nu}\operatorname{sgn}(x-t)\,K_1\left(\frac{U|x-t|}{2\nu}\right)\right]dt=\frac{a_1\lambda U}{2\nu}F_+(x)+u_-(x)$$
$$\text{for all } x. \tag{3.4}$$

Equation (3.4) is in the form needed for application of the Fourier transform.

We define

$$\mathrm{F}[g(x)]=\bar{g}(w)=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^\infty e^{ixw}g(x)\,dx, \tag{3.5}$$

and in general $\bar{g}(w)$ will be regular in some interval $\tau_- < \tau < \tau_+$, where $w=\sigma+i\tau$. Applying (3.5) to the integral equation (3.4), and using the convolution theorem

$$\int_{-\infty}^\infty e^{-ixw}\bar{f}(w)\,\bar{g}(w)\,dw=\int_{-\infty}^\infty f(t)\,g(x-t)\,dt,$$

we get

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^\infty U_+\exp\left\{-\frac{Ux}{2\nu}+ixw\right\}dx+\frac{1}{(2\pi)^{\frac{1}{2}}}\bar{F}_+(w)\,\mathrm{F}\left[\frac{1}{x}\exp\left\{-\frac{Ux}{2\nu}-\alpha|x|\right\}\right.$$
$$\left.-\frac{U}{2\nu}K_0\left(\frac{U|x|}{2\nu}\right)-\operatorname{sgn}(x)\frac{U}{2\nu}K_1\left(\frac{U|x|}{2\nu}\right)\right]=\frac{a_1\lambda U}{2\nu}\bar{F}_+(w)+\bar{u}_-(w). \tag{3.6}$$

The values of the various transforms involved in (3.6) and their ranges of existence are as follows:

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^\infty U\exp\left\{-\frac{Ux}{2\nu}+ixw\right\}dx=-\frac{U}{(2\pi)^{\frac{1}{2}}}\left(iw-\frac{U}{2\nu}\right)^{-1}\quad\left(\tau>-\frac{U}{2\nu}\right); \tag{3.7}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{x}\exp\left\{x\left(iw-\frac{U}{2\nu}\right)-\alpha|x|\right\}dx = \frac{i}{\pi}\tan^{-1}\left\{\left(w+\frac{iU}{2\nu}\right)\Big/\alpha\right\} \quad (\alpha > 0)$$

$$\left(\alpha > \tau+\frac{U}{2\nu} > -\alpha\right); \tag{3.8}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{U}{2\nu}K_0\left(\frac{U|x|}{2\nu}\right)e^{ixw}dx = -\frac{1}{2}\frac{U}{2\nu}\left[\left(\frac{U}{2\nu}\right)^2+w^2\right]^{-\frac{1}{2}} \quad \left(\frac{U}{2\nu} > \tau > -\frac{U}{2\nu}\right); \tag{3.9}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\operatorname{sgn}(x)\frac{U}{2\nu}K_1\left(\frac{U|x|}{2\nu}\right)e^{ixw}dx = -\tfrac{1}{2}iw\left[\left(\frac{U}{2\nu}\right)^2+w^2\right]^{-\frac{1}{2}} \quad \left(\frac{U}{2\nu} > \tau > -\frac{U}{2\nu}\right). \tag{3.10}$$

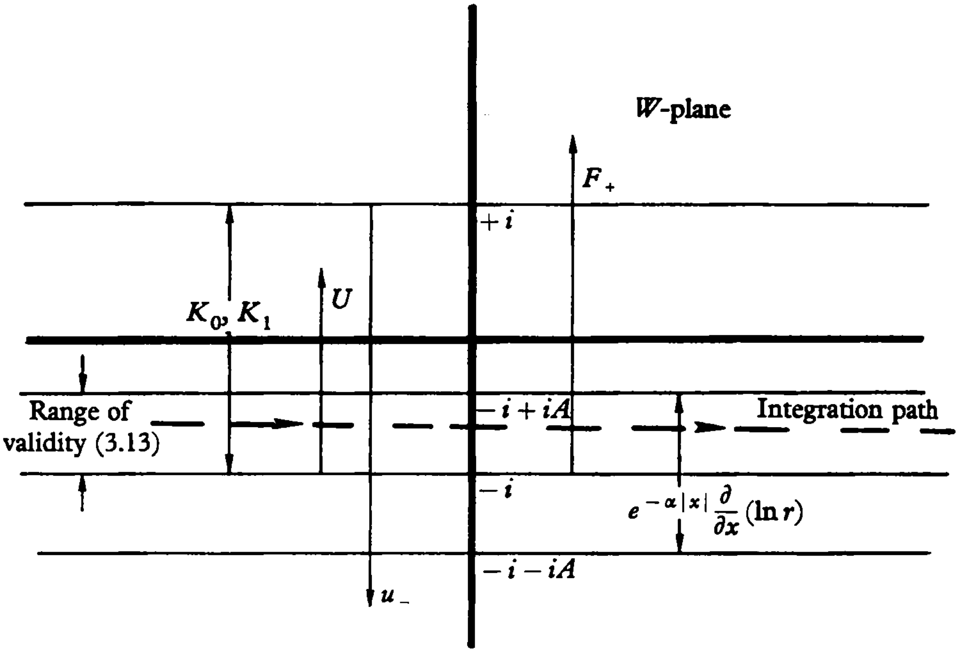

FIGURE 1. Regions of analyticity of the terms of the transformed equation.

Substituting these results into (3.6) we get

$$\bar{F}_+(w)\left[\frac{2}{\pi}\tan^{-1}\left\{\left(w+\frac{iU}{2\nu}\right)\Big/\alpha\right\}+i\frac{a_1\lambda U}{\nu}-\left\{\left(w-\frac{iU}{2\nu}\right)\Big/\left(w+\frac{iU}{2\nu}\right)\right\}^{\frac{1}{2}}\right]$$

$$= \sqrt{\frac{2}{\pi}}\,U\left(w+\frac{iU}{2\nu}\right)^{-1}-2i\bar{u}_-(w). \tag{3.11}$$

Finally, if we put†

$$W = \frac{2\nu}{U}w, \quad \Lambda = a_1\frac{U\lambda}{\nu}, \quad A = \frac{U\alpha}{2\nu}, \tag{3.12}$$

then
$$\bar{F}_+(W)\left[\frac{2}{\pi}\tan^{-1}\frac{W+i}{A}+i\Lambda-\left(\frac{W-i}{W+i}\right)^{\frac{1}{2}}\right] = \sqrt{\frac{2}{\pi}}\frac{2\nu}{W+i}-2i\bar{u}_-(W), \tag{3.13}$$

and this equation is to be solved for $\bar{F}_+(W)$.

† Note that Λ is proportional to both the Reynolds number based on the mean free path and, since $\lambda = (\pi\gamma/2)^{\frac{1}{2}}\,\nu/a$, to the Mach number $M = U/a$, where a is the velocity of sound and γ the ratio of the specific heats.

The ranges of existence of the various terms in (3.13) are indicated in figure 1. It is assumed that $f(x)$ is bounded for $x \to \infty$, so that $F_+(x) \sim e^{-Ux/2\nu}$ and $\bar{F}_+(w)$ exists for $\tau > -iU/2\nu$. Clearly $u \to 0$ as $x \to -\infty$, so that $u_- \sim e^{Ux/2\nu}$, $x \to -\infty$, and hence $\bar{u}_-(w)$ converges for $\tau < iU/2\nu$. The former assumption is based on the fact that $f(x)$ is proportional to the shearing stress on the plate (equation (2.8)), and since the slip solution must reduce to the zero-slip case far downstream, it follows that $f(x) \to 0$ as $x \to \infty$. We use $\mathscr{I}(W)$ for imaginary part of W and from the ranges of validity of the terms of equation (3.13), as depicted in figure 1, we see that the transformed equation holds where

$$-1+A > \mathscr{I}(W) > -1. \tag{3.14}$$

4. Resolution of the transformed equation

To solve (3.13) for $\bar{F}_+(W)$ it is necessary to write it as the sum of terms analytic in the upper and lower half planes. To this end we need to write

$$K(W) = \frac{2}{\pi}\tan^{-1}\frac{W+i}{A} + i\Lambda - \left(\frac{W-i}{W+i}\right)^{\frac{1}{2}} \tag{4.1}$$

as

$$K(W) = \frac{K_+(W)}{K_-(W)}, \tag{4.2}$$

where $K_+(W)$ is analytic for $\mathscr{I}(W) > -1$ and $K_-(W)$ is analytic for

$$\mathscr{I}(W) < -1+A.$$

There is a standard method for performing this factorization (see Noble 1958, p. 21). Thus we must consider

$$h(W) = \frac{d}{dW}[\ln K(W)] = \frac{d}{dW}\ln K_+(W) - \frac{d}{dW}\ln K_-(W)$$
$$= \frac{(2/\pi)\{A/W'^2+A^2)\} - i(W'-2i)^{-\frac{1}{2}}W'^{-\frac{3}{2}}}{(2/\pi)\tan^{-1}(W'/A) - i\Lambda - (W'-2i)^{\frac{1}{2}}W'^{-\frac{1}{2}}}, \tag{4.3}$$

where $W' = W+i$. Application of Cauchy's theorem to a rectangular region lying within the strip of regularity of $h(W)$, $a > \mathscr{I}(W) > b$, yields

$$h(W) = \frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{h(s)}{s-W}ds - \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{h(s)}{s-W}ds,$$

where the first and second members of the right-hand side are regular in $\mathscr{I}(W) > b$, $\mathscr{I}(W) < a$, respectively. Hence we have

$$\frac{d}{dW}\ln K_+(W) = \frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{h(s)}{s-W}ds, \quad \frac{d}{dW}\ln K_-(W) = \frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\frac{h(s)}{s-W}ds. \tag{4.4}$$

The strip of regularity of $h(W)$, $b < \mathscr{I}(W) < a$, in fact coincides with the region of analyticity of $K(W)$, $-1 < \mathscr{I}(W) < -1+A$, since by suitable choice of the branches of $(W+i)^{\frac{1}{2}}$ and $(W-i)^{\frac{1}{2}}$ we can ensure that the denominator of $h(W)$ in equation (4.3) is non-zero everywhere.

The singular points of $h(W)$ are at $W' = \pm iA$, $2i$, 0, and the paths of integration for the integrals in (4.4) lie between $W' = 0$ and $W' = iA$, as seen in figure 1. The contour integral for $(d/dW')\ln K_+(W')$ can be deformed into a path along the cut from $W' = 0$ to $\mathscr{I}(W') = -\infty$, figure 2. After reduction in the limit $A \to 0$, this integral becomes

$$\frac{d}{dW}\{\ln K_+(W)\} = -\frac{1}{2(W+i)} + \frac{\Lambda}{\pi}\int_1^\infty \frac{2(r^2-1)^{\frac{1}{2}} + (2+\Lambda^2)r - \Lambda^2}{\Lambda^2(\Lambda^2+4)r^2 - 2(2+\Lambda^2)\Lambda^2 r + 4 + \Lambda^4}\,\frac{dr}{(r^2-1)^{\frac{1}{2}}(ir+W)}. \tag{4.5}$$

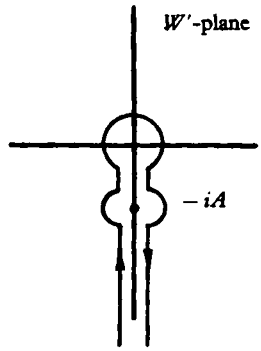

FIGURE 2. Contour integral for resolution of the transformed equation.

The roots of the quadratic form in the denominator of the integrand are given by

$$\alpha_1, \alpha_2 = \frac{2+\Lambda^2}{4+\Lambda^2} \pm \frac{4i}{\Lambda(4+\Lambda^2)}, \tag{4.6}$$

and in terms of these parameters we find that (4.5) yields

$$\frac{d}{dW}\{\ln K_+(W)\} = -\frac{1}{2(W+i)} + \frac{1}{4\pi i}\,\frac{1}{(W+i\alpha_1)(W+i\alpha_2)}\left[\frac{8}{\Lambda(4+\Lambda^2)}\ln\{\tfrac{1}{2}\Lambda^2(1-iW)\} + 2\pi\left(iW - \frac{2+\Lambda^2}{4+\Lambda^2}\right) - \frac{4i}{\Lambda(4+\Lambda^2)}\,\frac{(2+\Lambda^2)iW-\Lambda^2}{(1+W^2)^{\frac{1}{2}}}\ln\{-[iW+i(1+W^2)^{\frac{1}{2}}]\}\right]. \tag{4.7}$$

The integrated form of (4.7) is

$$K_+(W) = (1-iW)^{-\frac{1}{2}}\exp\left[-\frac{1}{\pi}\,\frac{2}{\Lambda(4+\Lambda^2)}\int_{-1}^{-iW}\frac{\ln\frac{1}{2}\Lambda^2(1+t)}{(t+\alpha_1)(t+\alpha_2)}\,dt\right] \times [-\Lambda^2(4+\Lambda^2)W^2 - 2i\Lambda^2(2+\Lambda^2)W + 4 + \Lambda^4]^{\frac{1}{2}} \times \exp\left[-\frac{1}{\pi}\,\frac{1}{\Lambda(4+\Lambda^2)}\int_{i\pi}^{\cosh^{-1}-iW}\frac{(2+\Lambda^2)\cosh\theta+\Lambda^2}{(\cosh\theta+\alpha_1)(\cosh\theta+\alpha_2)}\,d\theta\right]. \tag{4.8}$$

A similar integration can be carried out to find $K_-(W)$, or more simply, it can be found using the relation

$$K_-(W) = \frac{K_+(W)}{K(W)}.$$

Once $K(W)$ has been resolved into parts regular in the upper and lower half planes, we can write the transformed equation (3.13) as

$$\bar{F}_+(W)\,K_+(W) - \sqrt{\frac{2}{\pi}}\,2\nu\frac{K_-(-i)}{W+i} = \sqrt{\frac{2}{\pi}}\,2\nu\frac{K_-(W)-K_-(-i)}{W+i} - 2i\bar{u}_-(W)\,K_-(W), \tag{4.9}$$

in which the left-hand side is regular for $\mathscr{I}(W) > -1$ and the right-hand side is regular for $\mathscr{I}(W) < -1+A$. Hence, both sides having the common region of validity, $-1 < \mathscr{I}(W) < -1+A$, each must be equal to the same entire function of W. Moreover, $K_-(W) \to$ constant as $W \to \infty$, and, since $u_-(x)$ approaches a constant value as $x \to 0$, $\bar{u}_-(W) \sim 1/W$ for $W \to \infty$; hence the right-hand side of (4.9) approaches zero algebraically as $W \to \infty$, and the entire function represented by (4.9) must in fact be zero. Thus finally we have

$$\bar{F}_+(W) = \sqrt{\frac{2}{\pi}}\,2\nu\,\frac{K_-(-i)}{(W+i)\,K_+(W)} \tag{4.10}$$

and
$$\begin{aligned} F_+(x) &= \frac{U}{\pi}\,K_-(-i)\int_{-\infty}^{\infty}\frac{\exp(-iUxW/2\nu)}{(W+i)\,K_+(W)}\,dW \\ &= \frac{U}{\pi}\left[-\left(\frac{W+i}{-2i}\right)^{\frac{1}{2}}K_+(W)\right]_{W=-i}\int_{-\infty}^{\infty}\frac{\exp(-iUxW/2\nu)}{(W+i)\,K_+(W)}\,dW, \end{aligned} \tag{4.11}$$

since, from (4.1) and (4.2),

$$K_-(-i) = \lim_{W\to -i}\left[-K_+(W)\left(\frac{W+i}{-2i}\right)^{\frac{1}{2}}\right].$$

Equation (4.11) represents the full solution of the integral equation and hence of the problem. However the integration indicated in (4.11) cannot, in general, be carried out, and therefore it becomes necessary to study approximate forms of the solution for particular ranges of the variables. Thus, it is possible to obtain expressions valid for small and large values of x, using Tauberian theorems that relate a function and its transform for large and small values of its argument.

5. Asymptotic forms of the solution

We obtain an approximation to $F_+(x)$ for large positive x by studying the expansion of $\bar{F}_+(W)$ about its singularity with largest imaginary part (cf. Carslaw & Jaeger 1947, p. 279). This is at $W = -i$, i.e. at $W' = 0$. From the solution (4.8) for $K_+(W)$, we can derive the expansion

$$\begin{aligned} \frac{1}{W'K_+(W')} &= \frac{1}{\sqrt{2}}(-iW')^{-\frac{1}{2}} \\ &\times\left[1-\frac{\Lambda}{\sqrt{2}}(-iW')^{\frac{1}{2}}-\frac{\Lambda}{2\pi}iW'\ln(-\tfrac{1}{2}\Lambda^2 iW')-\tfrac{1}{2}\Lambda^2 iW'+\ldots\right] \end{aligned} \tag{5.1}$$

for small W'. So, from (4.11), we get

$$F_+(x) = -\frac{U}{\pi}\,e^{-Ux/2\nu}\int_{-\infty}^{\infty}\frac{\exp(-iUxW'/2\nu)}{W'K_+(W')}\,dW'. \tag{5.2}$$

There are a number of Tauberian theorems that relate a transformed expansion to its original (see Doetsch 1937, p. 265), but because of the mixed nature of the terms in (5.1) none of these theorems applies directly. However, they can be used formally, and the result is

$$F_+(x) \sim \frac{2U}{\sqrt{\pi}}\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}} e^{-Ux/2\nu}\left[1-\frac{\Lambda^2}{2}\left(\frac{\nu}{Ux}\right)+\frac{\Lambda}{2\pi}\left(\ln\frac{Ux}{2\nu}-2+2\ln 2+C\right)\frac{\nu}{Ux}+\dots\right], \tag{5.3}$$

where C = Euler's constant = 0·577 ..., for $x \to \infty$. That (5.3) is indeed the correct expansion can be proved directly from (5.2) (Carlsaw & Jaeger 1947, p. 280). The procedure is to integrate along a contour of the type indicated in figure 3, in which the horizontal path is a distance $\epsilon_1 < \Lambda$ above the real axis, and we choose the radius of the circular path about the singularity at $W' = 0$ to be ϵ_2/x. In the limit $x \to \infty$, the integrals along the horizontal portions and about the circle vanish asymptotically and the vertical path yields the contribution written in (5.3). This method may be applied to obtain any further number of terms in the asymptotic expansion for large x.

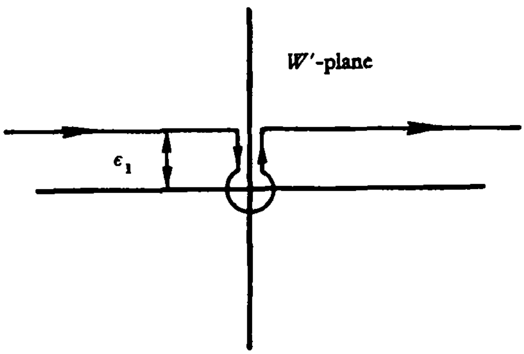

FIGURE 3. Integration path for the asymptotically large x expansion.

The other expansion required is for small positive x and this we obtain by investigating the transformed function for large W. We will evaluate the leading term only.† From (4.8) we get, for $|W| \to \infty$, $\mathscr{I}(W) > 0$,

$$K_+(W) \sim \frac{1}{i}[\Lambda^2(4+\Lambda^2)]^{\frac{1}{4}} \exp\left[-\frac{1}{\pi}\frac{2}{\Lambda(4+\Lambda^2)}\int_{-1}^{\infty}\frac{\ln\frac{1}{2}\Lambda^2(1+t)}{(t+\alpha_1)(t+\alpha_2)}dt\right]$$
$$\times\exp\left[-\frac{1}{\pi}\frac{1}{\Lambda(4+\Lambda^2)}\int_{i\pi}^{\infty}\frac{(2+\Lambda^2)\cosh\theta+\Lambda^2}{(\cosh\theta+\alpha_1)(\cosh\theta+\alpha_2)}d\theta\right]. \tag{5.4}$$

Moreover, we find on integration that

$$\exp\left[-\frac{1}{\pi}\frac{2}{\Lambda(4+\Lambda^2)}\int_{-1}^{\infty}\frac{\ln\frac{1}{2}\Lambda^2(1+t)}{(t+\alpha_1)(t+\alpha_2)}dt\right] \sim \left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{[\tan^{-1}(2/\Lambda)-\pi]/2\pi}$$

† Closer study shows that difficulties may occur in obtaining further terms of the series if Λ is large (the boundary-layer limit), due to non-uniformities when $W \to \infty$, $\Lambda \to \infty$.

and

$$\exp\left[-\frac{1}{\pi}\frac{1}{\Lambda(4+\Lambda^2)}\int_{i\pi}^{\infty}\frac{(2+\Lambda^2)\cosh\theta+\Lambda^2}{(\cosh\theta+\alpha_1)(\cosh\theta+\alpha_2)}d\theta\right]$$

$$\sim\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{[\pi-\tan^{-1}(2/\Lambda)]/2\pi}\exp\left\{\frac{1}{2\pi i}\left[L_2\left(\frac{\Lambda}{\Lambda+2i}\right)-L_2\left(\frac{\Lambda}{\Lambda-2i}\right)\right]\right\}$$

$$=\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{[\pi-\tan^{-1}(2/\Lambda)]/2\pi}\exp\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n\sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right],$$

where $L_2(z)$ is the dilogarithmic function

$$L_2(z)=\int_0^z\frac{\ln s}{1-s}ds.$$

Hence, altogether, in the limit $|W|\rightarrow\infty$, we have

$$K_+(W)\sim-i\Lambda\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{-\frac{1}{2}}\exp\left[-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n\sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right]. \quad (5.5)$$

Using the inversion formula (5.2) and the appropriate Tauberian theorem (Doetsch 1937, p. 269), (5.5) yields the result†

$$F_+(0)=2U\Lambda^{-1}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\exp\left[-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n\sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right]. \quad (5.6)$$

Finally, we can obtain an expansion of $F_+(x)$ valid for small values of Λ. Thus, from (4.8), we obtain the following expression for $\Lambda\rightarrow 0$,

$$K_+(W)\simeq(2\Lambda)^{\frac{1}{2}}\frac{\{(1/\Lambda^2)-W^2\}^{\frac{1}{4}}}{(1-iW)^{\frac{1}{2}}}\exp\left\{\frac{1}{8\pi}\int_0^{-4\tan^{-1}i\Lambda W}\ln(2\sin\tfrac{1}{2}\theta)\,d\theta\right\}.$$

Substituting into (5.2) this gives

$$F_+(x)\simeq\frac{U}{\pi}\frac{1}{(2\Lambda)^{\frac{1}{2}}}\int_{-i\infty}^{i\infty}\frac{i\exp\left[\dfrac{Ux\xi}{2\nu}-\dfrac{1}{8\pi}\displaystyle\int_0^{4\tan^{-1}\Lambda\xi}\ln(2\sin\tfrac{1}{2}\theta)\,d\theta\right]}{\xi^{\frac{1}{2}}\{\xi^2+(1/\Lambda^2)\}^{\frac{1}{4}}}d\xi \quad (5.7)$$

approximately, where $\xi=iW$. The path of integration lies to the right of the singularity of the integrand at $\xi=0$. Numerical integration of (5.7) has been carried out for a series of values of the relevant parameter $Ux/2\nu\Lambda=x/(a_1\lambda)$, and the results will be discussed in the next section.

6. Shearing force and slip velocity

The approximate expressions developed in §5 enable us to find the plate shearing stress (and hence the slip velocity) directly without further integration, using the result given in §2; namely that

$$\left|\frac{\partial u}{\partial y}\right|=\frac{U}{2\nu}F(x)\,e^{Ux/2\nu}\quad(y=0,x>0). \quad (6.1)$$

† We remark here that the exponential term in (5.6) is never far from unity, and, to within approximately 5 %, $F_+(0)=2U\Lambda^{-1}\{\Lambda/(4+\Lambda^2)^{\frac{1}{2}}\}^{\frac{1}{2}}$.

First, for $x \to \infty$, we get from (5.3)

$$\left|\frac{\partial u}{\partial y}\right| \sim \frac{U^2}{\sqrt{\pi}\,\nu}\left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}\left[1-\frac{a_1^2}{2}\left(\frac{U\lambda^2}{\nu x}\right)+\frac{a_1\lambda}{2\pi x}\left(\ln\frac{Ux}{2\nu}-0{\cdot}04\right)+\dots\right], \tag{6.2}$$

in which we have replaced Λ by $a_1(U\lambda/\nu)$, and have evaluated the numerical coefficient in the last term. Defining

$$c_f = \frac{\nu\,|\partial u/\partial y|}{\frac{1}{2}U^2},$$

we get
$$c_f \sim \frac{2}{\sqrt{\pi}}\left(\frac{\nu}{Ux}\right)^{\frac{1}{2}}\left[1-\frac{a_1^2}{2}\left(\frac{U\lambda^2}{\nu x}\right)+\frac{a_1\lambda}{2\pi x}\left(\ln\frac{Ux}{2\nu}-0{\cdot}04\right)+\dots\right]. \tag{6.3}$$

Slip boundary-layer theory gives the following expression for the skin-friction coefficient (Mirels 1952)

$$c_{f_{bl}} = \frac{2}{\sqrt{\pi}}\left(\frac{\nu}{Ux}\right)^{\frac{1}{2}}\left[1-\frac{a_1^2}{2}\left(\frac{U\lambda^2}{\nu x}\right)+O\left(\frac{U\lambda^2}{\nu x}\right)^2\right], \tag{6.4}$$

so that the full Oseen equations yield a correction to the zero-slip skin-friction coefficient of the first order in λ, whereas the lowest-order term in the boundary-layer result is of order λ^2. In fact the former term is dominant for $U\lambda/\nu \ll 1$, but becomes negligible for $U\lambda/\nu \to \infty$. Since $\lambda \propto M/(U/\nu)$, where M is the Mach number of the free-stream flow, an alternative statement is that the additional terms arising from use of the full equations of motion are important in estimating the plate shearing stress for small Mach numbers, while for $M \to \infty$ boundary-layer theory gives the correct value.† Within the range of applicability of the present analysis, since we are treating the incompressible case (low Mach number), it therefore follows that boundary-layer analysis does not give a satisfactory estimate of the shearing stress, even far downstream, where slip effects are small (but not negligible). Since the slip condition is $u' = \lambda(\partial u/\partial y)$, $y = 0$, similar statements apply to values of the velocity at the plate surface.

For $x \to 0$ we have the result (equations (5.6) and (6.1))

$$u+U = a_1\lambda\left|\frac{\partial u}{\partial y}\right| \to U\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\exp\left[-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n\sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right] \tag{6.5}$$

and
$$c_f \to \frac{2}{\Lambda}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}\exp\left[-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n\sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right]. \tag{6.6}$$

For $\Lambda \to \infty$, (6.5) yields

$$U+u = U$$

in agreement with boundary-layer theory (Mirels 1952), and for $\Lambda \to 0$

$$U+u \to U(\tfrac{1}{2}\Lambda)^{\frac{1}{2}}, \quad c_f \to U(\tfrac{1}{2}\Lambda)^{-\frac{1}{2}}, \tag{6.7}$$

so that in the limit of zero slip we regain the normal result, namely $c_f \to \infty$ as $x \to 0$.

† The general conclusion, that with a slip-boundary condition the boundary-layer equations become valid in the limit M or $U\lambda/\nu \to \infty$, can in fact be deduced from the complete Oseen equations (2.1).

For small Λ (i.e. for small values of the Mach number M) we can present results valid for all x. Thus from (5.7), using (6.1), we obtain the approximate expression for c_f:

$$c_f = \frac{1}{\pi(2\Lambda)^{\frac{1}{2}}}\int_{-i\infty}^{i\infty} \frac{i\exp\left\{-\dfrac{1}{8\pi}\displaystyle\int_0^{4\tan^{-1}\Lambda\xi} \ln(2\sin\tfrac{1}{2}\theta)\,d\theta\right\}}{\xi^{\frac{1}{2}}\{\xi^2+(1/\Lambda^2)\}^{\frac{1}{4}}}\exp\left(-\frac{Ux\xi}{2\nu}\right)d\xi. \quad (6.8)$$

The relationship (6.8) between $\pi(\frac{1}{2}\Lambda)^{\frac{1}{2}}c_f$ and $Ux/2\nu\Lambda = x/2a_1\lambda$ evaluated by means of a high-speed electronic computer is plotted in figure 4. Shown also is the zero-slip result, for which $\pi(\frac{1}{2}\Lambda)^{\frac{1}{2}}c_f = \sqrt{\pi}(x/2a_1\lambda)^{-\frac{1}{2}}$. Increase of the shear above the zero-slip value for $x/2a_1\lambda$ greater than about 0·3 should be interpreted as a result of viscous interaction effects of order higher than included in boundary-layer theory rather than as a direct consequence of slip. The boundary-layer result is approached as we let Λ (or the Mach number) approach infinity, and in this limit the shear force is always less than the no-slip shear.

7. Displacement thickness

The total displacement thickness

$$\delta^* = -\int_0^\infty \frac{u}{U}dy \quad (7.1)$$

is infinite in the case of any semi-infinite body. However, we can find the displacement thickness corresponding to the transverse wave part of the solution $\mathbf{v}_2+\mathbf{v}_2^*$ (equation (2.5)), and in fact this gives a boundary-layer profile, the potential flow about which yields the longitudinal wave $\mathbf{v}_1$. We should make it clear that the boundary layer calculated in this way is not a line of demarcation between potential and rotational viscous flow, since at the leading edge, near or on the plate, neither potential nor vorticity waves dominate. Moreover, as will be shown, this displacement thickness is zero ahead of the plate, while in fact there is a spreading of vorticity upstream of the plate, as indicated by the fundamental solution $\mathbf{v}_2$ for the vorticity (equation (2.5)).

From (2.5) and (2.6) we have for the displacement thickness defined in this way

$$\delta^* = \frac{e^{Ux/2\nu}}{2\pi U}\int_0^\infty F(t)\int_0^\infty\left[\frac{U}{2\nu}K_0\left(\frac{Ur_t}{2\nu}\right)-\frac{U}{2\nu}\frac{x-t}{r_t}K_1\left(\frac{Ur_t}{2\nu}\right)\right]dy\,dt, \quad (7.2)$$

where $\quad r_t = \{(x-t)^2+y^2\}^{\frac{1}{2}},$

or
$$\delta^* = \frac{e^{Ux/2\nu}}{2\pi U}\int_0^\infty F(t)\left\{\frac{U}{2\nu}\left[\left(\frac{\nu|x-t|}{U}\right)^{\frac{1}{2}}\Gamma(\tfrac{1}{2})K_{-\frac{1}{2}}\left(\frac{U|x-t|}{2\nu}\right)\right.\right.$$
$$\left.\left.+(x-t)\left(\frac{\nu}{U|x-t|}\right)^{\frac{1}{2}}\Gamma(\tfrac{1}{2})K_{\frac{1}{2}}\left(\frac{U|x-t|}{2\nu}\right)\right]\right\}dt$$
$$= \frac{e^{Ux/2\nu}}{4U}\int_0^\infty F(t)\left[\exp\left\{-\frac{U|x-t|}{2\nu}\right\}+\operatorname{sgn}(x-t)\exp\left\{-\frac{U|x-t|}{2\nu}\right\}\right]dt. \quad (7.3)$$

For $x<0$ this is zero, while for $x>0$ we have

$$\delta^* = \frac{1}{4U}\int_0^x F(t)\,e^{Ut/2\nu}\,dt. \quad (7.4)$$

Asymptotic forms of δ^* for $x \to \infty$ and $x \to 0$ can be obtained using the results derived in §5. For $x \to \infty$ care has to be exercised in evaluating the integral, and to avoid convergence difficulties, we write

$$\begin{aligned}\int_0^x F(t)\,e^{Ut/2\nu}\,dt &= \int_0^x F(\infty)\,e^{Ut/2\nu}\,dt \\ &\quad + \int_0^\infty [F(t)-F(\infty)]\,e^{Ut/2\nu}\,dt - \int_x^\infty [F(t)-F(\infty)]\,e^{Ut/2\nu}\,dt \\ &= \int_0^x F(\infty)\,e^{Ut/2\nu}\,dt - \int_x^\infty [F(t)-F(\infty)]\,e^{Ut/2\nu}\,dt \\ &\quad + \lim_{w\to -iU/2\nu}\left[(2\pi)^{\frac{1}{2}}\bar{F}(w) - \int_0^\infty F(\infty)\,e^{itw}\,dt\right]. \end{aligned} \tag{7.5}$$

Using the expansion (5.3) this yields

$$\delta^*(x) \sim -a_1\lambda + \sqrt{\frac{2}{\pi}\frac{2\nu}{U}}\left(\frac{Ux}{2\nu}\right)^{\frac{1}{2}} \times\left\{1+\frac{a_1^2}{2}\left(\frac{U\lambda^2}{\nu x}\right) - \frac{a_1\lambda}{2\pi x}\left(\ln\frac{Ux}{2\nu} - 0{\cdot}04\right)+\ldots\right\} \quad (x\to\infty), \tag{7.6}$$

and far enough downstream this gives the usual boundary-layer result (Schaaf & Chambré 1957), namely that the boundary-layer thickness is a mean free path thinner than the zero-slip value.

The expansion for δ^* near $x = 0$ is obtained directly from (5.6):

$$\begin{aligned}\delta^* &\simeq \frac{1}{2U}\int_0^x 2U\Lambda^{-1}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \exp\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n \sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right]dt \\ &= \frac{x}{\Lambda}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \exp\left[\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{\frac{\Lambda}{(4+\Lambda^2)^{\frac{1}{2}}}\right\}^n \sin\left(n\tan^{-1}\frac{2}{\Lambda}\right)\right],\end{aligned} \tag{7.7}$$

again agreeing with boundary-layer theory as $\Lambda \to \infty$ (Schaaf & Chambré 1957).

8. Experimental evidence

As has been discussed in §1, it appears essential that experimental checks on slip-flow analyses be carried out under low subsonic incompressible-flow conditions. Moreover, the present theory has been shown to give results essentially different from past work, based on boundary-layer approximations, in the low Mach number range.

Unfortunately, very little experimental work has been carried out in this régime and to the author's knowledge the only relevant study is that carried out by Sherman (1952) in measurements of the drag of flat plates in a low-density subsonic airstream. To be able to compare our results with such experiments, an integration of the expression for the shearing stress on the plate has to be made. Such an integration has been carried out graphically for the case Λ small (low Mach number) from figure 4. The results are shown in figure 5 in terms of a

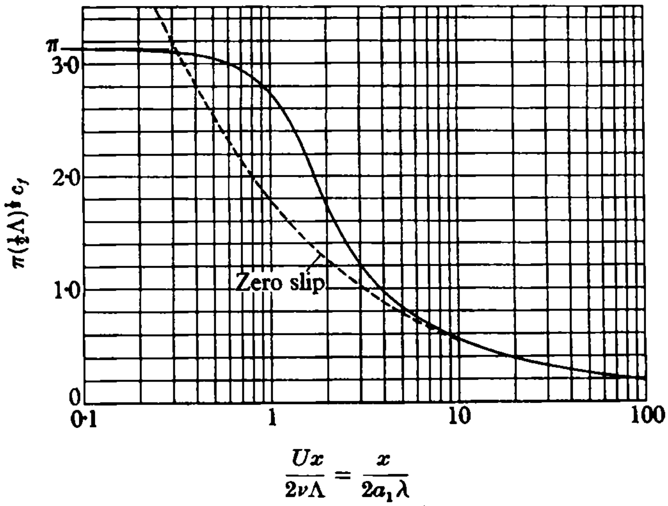

FIGURE 4. Skin-friction coefficient distribution for Λ small.

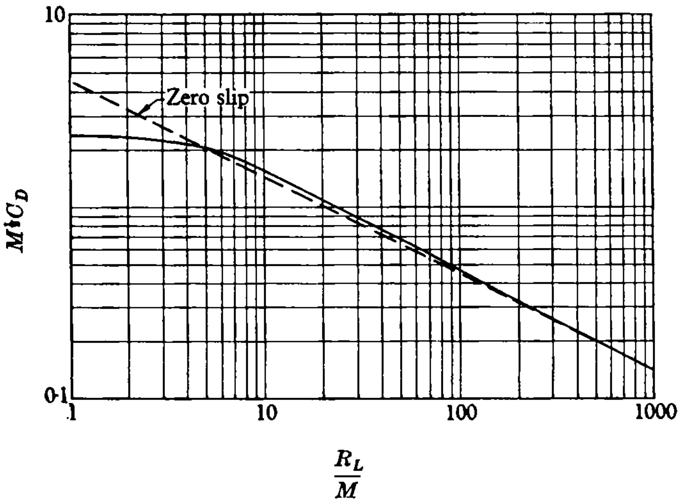

FIGURE 5. Drag coefficient variation for Λ small.

Reynolds number R_L based on an integration length from $x = 0$ to $x = L$ along the plate and a drag coefficient defined by

$$C_D = \frac{2}{L}\int_0^L c_f\, dx.$$

It will be seen from the figure that significant departures from the no-slip cases are not predicted until $R_L/M < 4$ approximately. In Sherman's experimental work the smallest value of R_L/M was 6 and thus marked effects of slip should not be visible in his low-speed data. In view of experimental errors, the corrections necessary to allow for compressibility and for the finite length of the plates (Kuo 1953) and for the approximations of linearized theory (Lewis & Carrier

1949), no observations as to the validity of this analysis or those of other slip-flow theories can be made. It is desirable and probably essential not only that experiments at lower Reynolds numbers be made, but that experiments on the local modifications of the flow due to slip (such as the change in plate shearing stress c_f) be undertaken in order that a definite conclusion be reached.

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